CDI-II – Prática 25/5/21

Ficha 11 + Ficha 12

Ficha 11 – $F = \nabla \varphi$

– F é o gradiente de $\varphi: \mathbb{R}^n \to \mathbb{R}$, C^1

$F: \mathbb{R}^n \to \mathbb{R}^n$. (conservativo)

φ = potencial escalar de F.

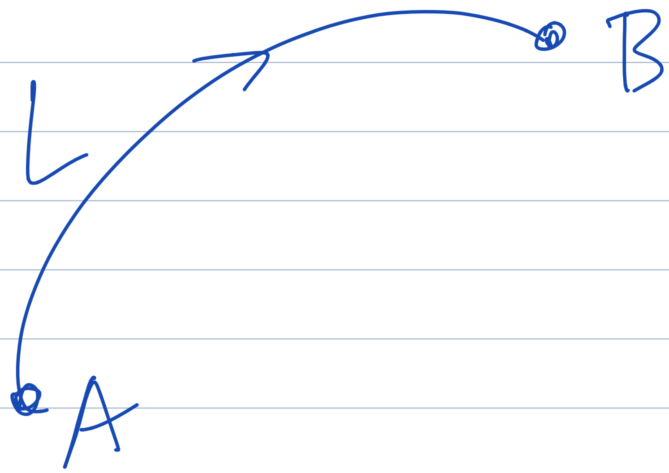
– Se $F = \nabla \varphi$ então F é fechado:

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j} \quad j \neq k$$

– $R(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

- TFC: Se $F = \nabla\varphi$ então

$$\int_L F = \varphi(B) - \varphi(A)$$



3-a) $\dfrac{\partial F_1}{\partial y} = 2y \qquad \dfrac{\partial F_2}{\partial x} = 3x^2$

$$\neq$$

$F = \alpha$ não é fechado $\Rightarrow$ não é conservativo. Não tem potencial

3-b) $b(x,y) = (x^3, y^2) + (y, x)$

$$\varphi(x,y) = \frac{x^4}{4} + \frac{y^3}{3} + xy + C$$

$C \in \mathbb{R}.$

3-c) $\varphi(x,y) = e^x + e^y + C$

3-d) $d(x,y) = \frac{1}{2}\left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$

$$\varphi(x,y) = \frac{1}{2}\log(x^2+y^2) + C$$

3-e) $1 = \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = 1$

$$0 = \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} = 0$$

$$0 = \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = 0$$

$F = e$ é fechado.

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = y \longrightarrow \varphi(x,y,z) = xy + \underbrace{A(y,z)}_{B(z)} \\ \dfrac{\partial \varphi}{\partial y} = x \longrightarrow \cancel{x} + \dfrac{\partial A}{\partial y} = \cancel{x} \rightarrow A(y,z) = B(z) \\ \dfrac{\partial \varphi}{\partial z} = 2z \longrightarrow B'(z) = 2z \end{cases}$$

$$B(z) = z^2 + C$$

$$\boxed{\varphi(x,y,z) = xy + z^2 + C}$$

$C \in \mathbb{R}$

$$e(x,y,z) = (y,x,0) + (0,0,2z)$$

$$\varphi(x,y,z) = xy + z^2 + C.$$

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3- f) $-1 = \frac{\partial g_1}{\partial y} \neq \frac{\partial g_2}{\partial x} = 1$

g não é fechado

$\Rightarrow$ g não é conservativo.

4 — $F = \nabla \varphi$

$$F(x,y,z) = \frac{1}{2}\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 4z\right)$$

$$F(x,y,z) = \frac{1}{2}\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 0\right) + (0,0,2z)$$

$$\boxed{\varphi(x,y,z) = \frac{1}{2}\log(1+x^2+y^2) + z^2 + C}$$

4 — a) $g(t) = (\cos t, \sin t, t)$

$0 \leq t \leq 2\pi$

$A = g(0) = (1,0,0)$, $B = g(2\pi)$

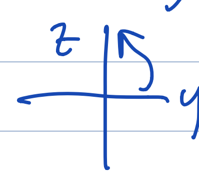
$= (1, 0, 2\pi)$

$$\varphi(A) = \varphi(1,0,0) = \frac{1}{2}\log 2 + C$$

$$\varphi(B) = \varphi(1,0,2\pi) = \frac{1}{2}\log 2 + 4\pi^2 + C$$

$$\varphi(B) - \varphi(A) = 4\pi^2 \ //.$$

4-b) $\begin{cases} y^2 + z^2 = 1 \\ x = y^2 - z^2 \end{cases}$ $\quad (\rho, \theta, x)$



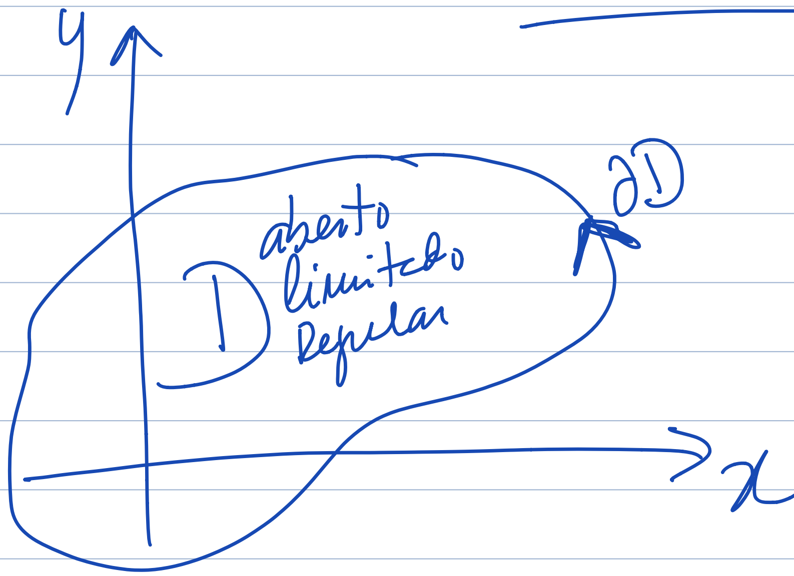
$y(t) = \cos t \quad ; \quad z(t) = \operatorname{sen} t$

$x(t) = \cos^2 t - \operatorname{sen}^2 t$

$g(t) = (\cos^2 t - \operatorname{sen}^2 t, \cos t, \operatorname{sen} t)$

$0 \leq t \leq 2\pi \qquad A = g(0) = g(2\pi) = B$

Ficha-12 (T. de Green ($\mathbb{R}^2$))



$F: \mathbb{R}^2 \to \mathbb{R}^2$

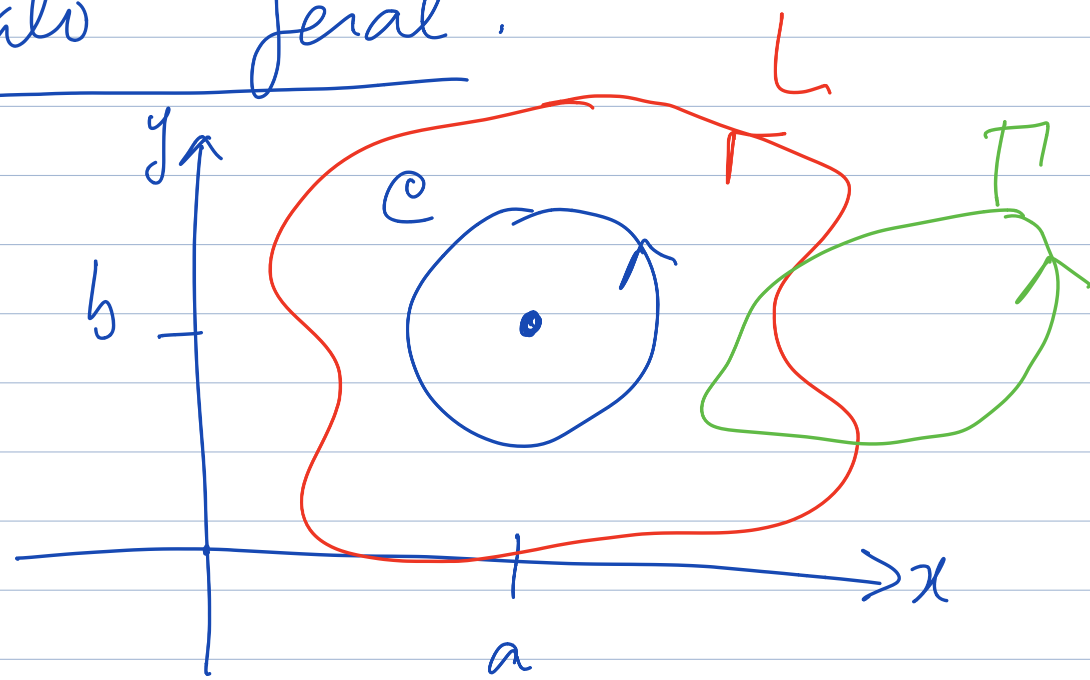
C^1 em D

$F = (P, Q)$

$$\underbrace{\oint_{\partial D} F \cdot dg}_{\text{Trabalho de } F \text{ na fronteira de } D} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

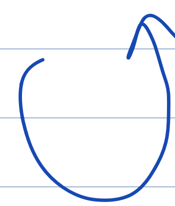
Trabalho de F na fronteira de D.

"Ralo" geral:



$$R(x,y) = \left(-\frac{y-b}{(x-a)^2+(y-b)^2}, \frac{x-a}{(x-a)^2+(y-b)^2} \right)$$

$$\mathbb{R}^2 \setminus \{(a,b)\}$$



Contas (definição) $\displaystyle\int_C R \cdot d\gamma = 2\pi$

Green: $\oint_L R\cdot dg = 2\pi$

$\oint_\Gamma R\cdot dg = 0$!!!

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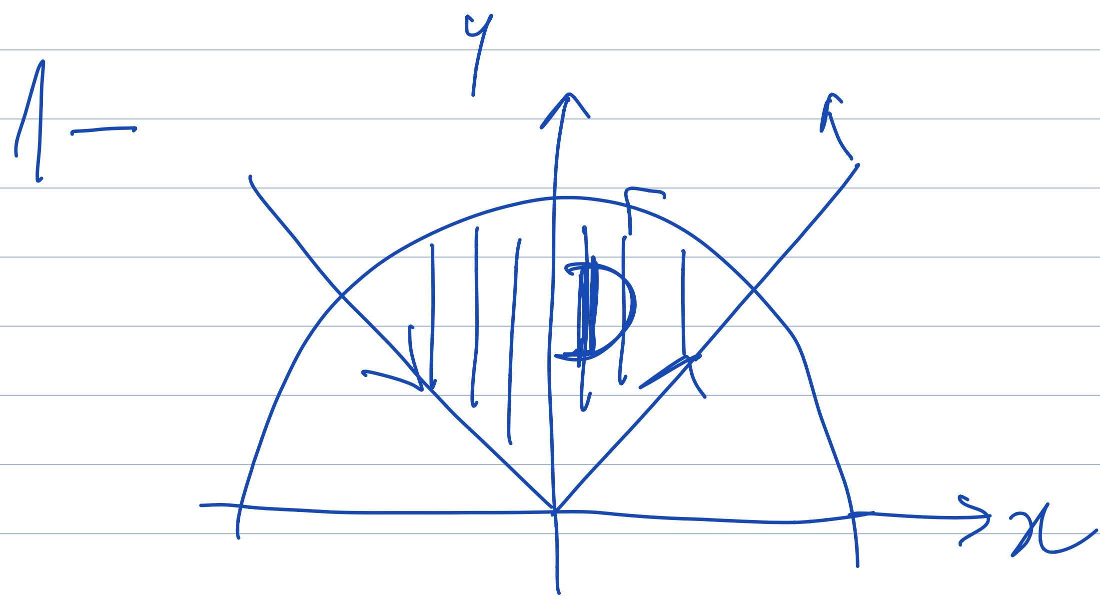

$$\int_{\partial D} F\cdot dg = \iint_{D} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{3} dx\,dy \qquad \circlearrowleft$$

$$Q(x,y) = x \quad ; \quad P(x,y) = -2y$$

$$\int_{\partial D} F\cdot dg = 3\, vol_2(D) = \frac{3\pi}{4} \quad //$$

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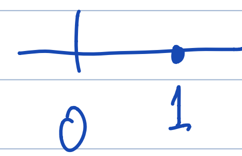
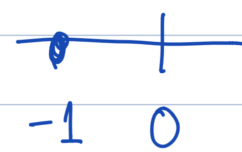
$$2-\quad F = G + H$$

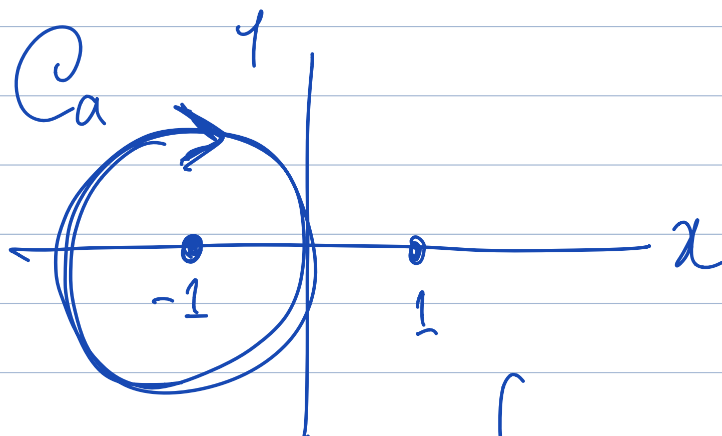
$$G(x,y) = \left(\frac{-y}{(x-1)^2+y^2} , \frac{x-1}{(x-1)^2+y^2} \right)$$



$$H(x,y) = \left(\frac{y}{(x+1)^2+y^2} , \frac{-(x+1)}{(x+1)^2+y^2} \right)$$



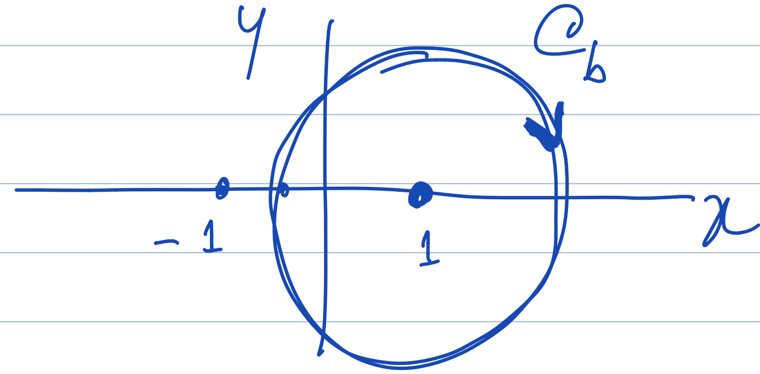
a)



$$\int_{C_a} G = 0 \qquad\qquad \int_{C_a} H = 2\pi$$

$$\Rightarrow \int_{C_a} F = 0 + 2\pi = 2\pi \quad //$$

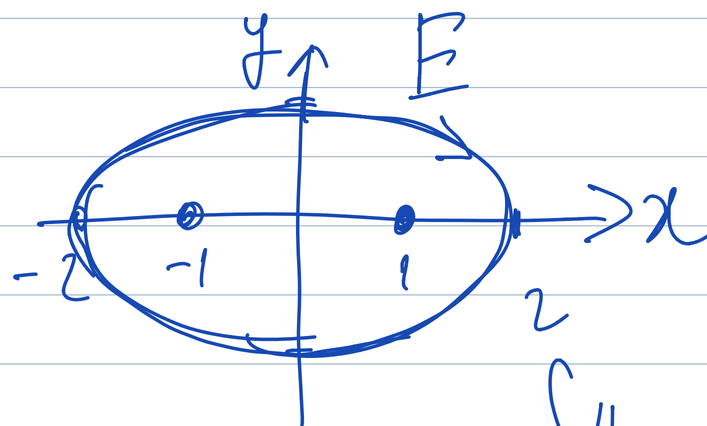
b)



$$\int_{C_b} H = 0 \qquad \int_{C_b} G = -2\pi$$

$$\int_{C_b} F = 0 - 2\pi = -2\pi$$ //

c)

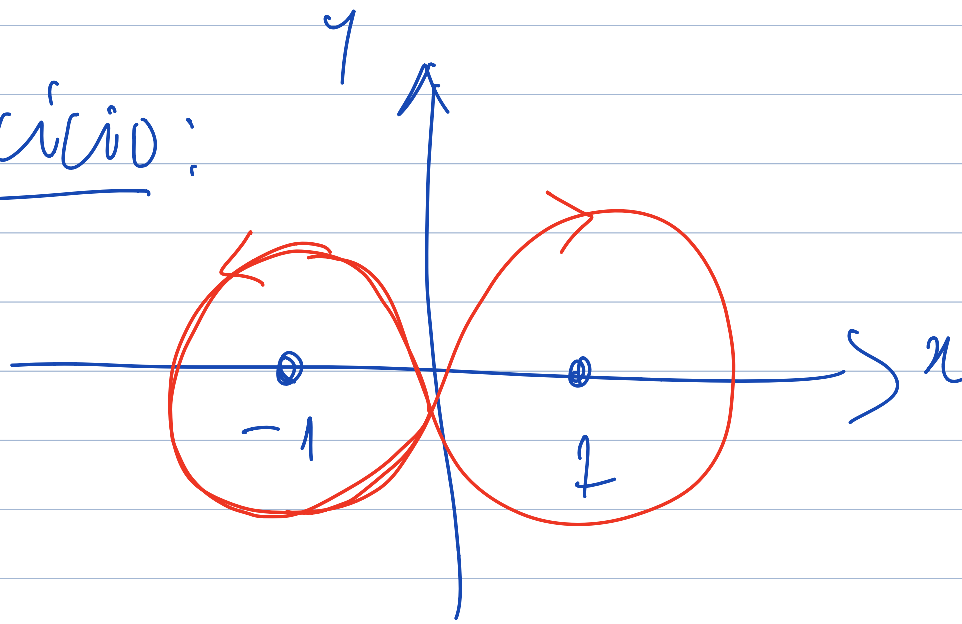


$$\int_E G = -2\pi \qquad \int_E H = 2\pi$$

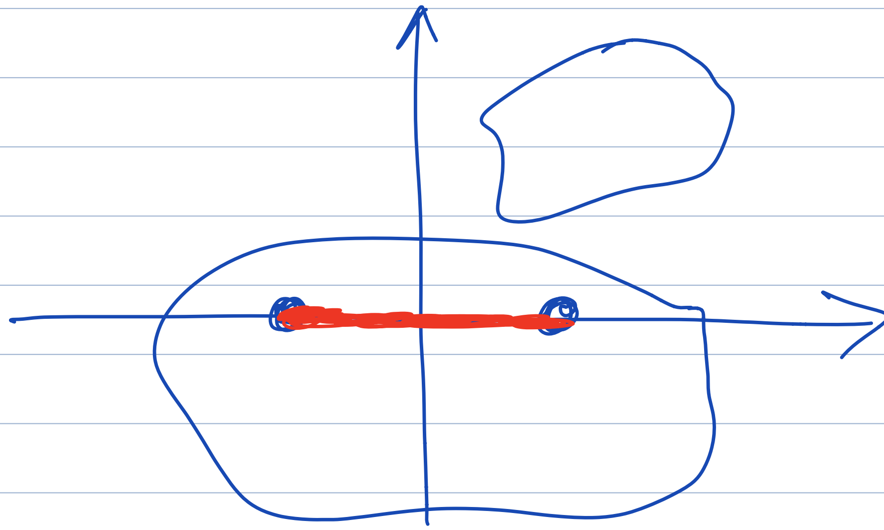
$$\int_E F = -2\pi + 2\pi = 0$$

F é gradiente sse $\oint_L F = 0$ para qualquer linha fechada L.

Exercício:



$$\mathbb{R}^2 \setminus \{(x,y) : y = 0\ ;\ |x| \leq 1\}$$



3-a) $F(x,y) = (-y, x)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1-(-1) = 2$$

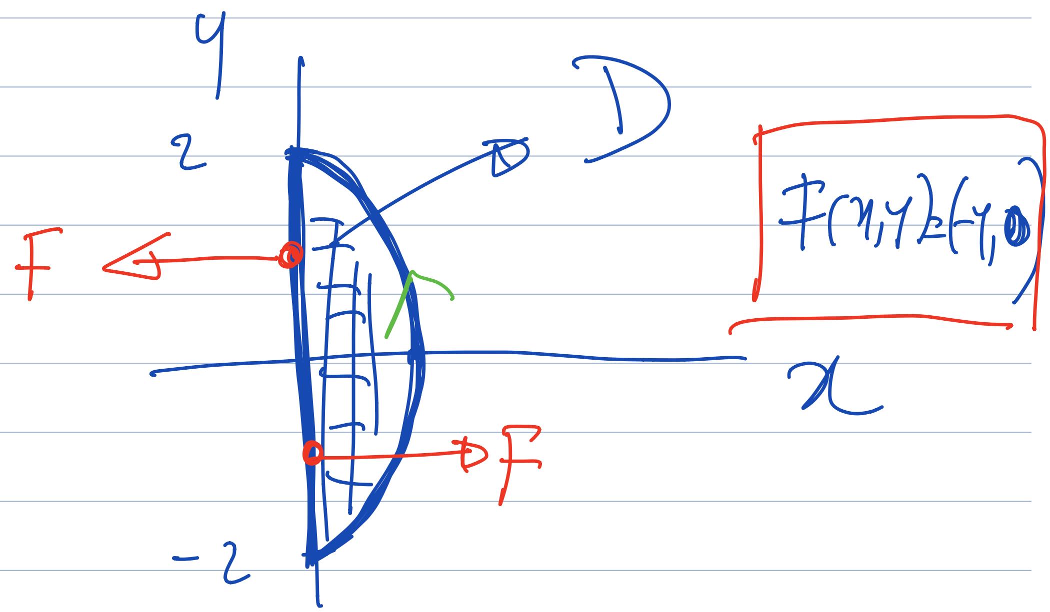
b) $F(x,y) = (-y, 0)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad //$$

c) $F(x,y) = (0, x)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad //$$

$$D: x^2 + \frac{y^2}{4} < 1, \; x > 0$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \qquad , \; F = (P, Q)$$

$$\mathrm{vol}_2(D) = \int_{\partial D} F$$

etc...